

ON THE ASSOCIATED SEQUENCES OF SPECIAL POLYNOMIALS

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ABSTRACT. In this paper, we investigate some properties of the associated sequence of Daehee and Changhee polynomials. Finally, we give some interesting identities of associated sequence involving some special polynomials.

1. Introduction

In this paper, we assume that $\lambda \in \mathbb{C}$, with $\lambda \neq 1$. For $\alpha \in \mathbb{R}$, the Frobenius-Euler polynomials of order α are defined by the generating function to be

$$(1) \quad \left(\frac{1-\lambda}{e^t-\lambda} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} H_n^{(\alpha)}(x|\lambda) \frac{t^n}{n!}, \quad (\text{see } [9, 11, 13, 22, 27]).$$

In the special case, $x = 0$, $H_n^{(\alpha)}(0|\lambda) = H_n^{(\alpha)}(\lambda)$ are called n -th Frobenius-Euler numbers of order α , (see [8, 9, 27]). As is well known, the Bernoulli polynomials of order α are defined by the generating function to be:

$$(2) \quad \left(\frac{t}{e^t-1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{t^n}{n!}, \quad (\text{see } [10, 12, 21, 28]).$$

In the special case, $x = 0$, $B_n^{(\alpha)}(0|\lambda) = B_n^{(\alpha)}$ are called n -th Bernoulli numbers of order α . The Stirling numbers of the second kind are defined by

$$(3) \quad (e^t - 1)^n = n! \sum_{l=n}^{\infty} S_2(l, n) \frac{t^l}{l!}, \quad (\text{see } [5, 20, 21, 26]).$$

and the Stirling numbers of the first kind are given by

$$(4) \quad (x)_n = x(x-1)\dots(x-n+1) = \sum_{l=0}^n S_1(l, n) x^l, \quad (\text{see } [5, 20, 21]).$$

Let \mathcal{F} be the set of all formal power series in the variable t over \mathbb{C} with

$$(5) \quad \mathcal{F} = \{f(t) = \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n \mid a_n \in \mathbb{C}\}.$$

Let us assume that \mathbb{P} is the algebra of polynomials in the variable x over \mathbb{C} and \mathbb{P}^* is the vector space of all linear functionals on \mathbb{P} . $\langle L \mid p(x) \rangle$ denotes the action of the linear functional L on a polynomial $p(x)$ and we remind that the vector space structure on \mathbb{P}^* is defined by $\langle L + M \mid p(x) \rangle =$

$\langle L \mid p(x) \rangle + \langle M \mid p(x) \rangle$, and $\langle cL \mid p(x) \rangle = c \langle L \mid p(x) \rangle$, where c is a complex constant, (see[6, 9, 20, 21]). For $f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \in \mathcal{F}$, we define a linear functional on \mathbb{P} by setting

$$(6) \quad \langle f(t) \mid x^n \rangle = a_n, (n \geq 0), \quad (\text{see}[20, 21]).$$

From (6), we note that

$$(7) \quad \langle t^k \mid x^n \rangle = n! \delta_{n,k}, \quad (n, k \geq 0),$$

where $\delta_{n,k}$ is the Kronecker's symbol. Let $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L \mid x^k \rangle}{k!} t^k$. Then, by (7), we see that $\langle f_L(t) \mid x^n \rangle = \langle L \mid x^n \rangle$. The map $L \mapsto f_L(t)$ is a vector space isomorphism from \mathbb{P}^* onto \mathcal{F} . Henceforth, \mathcal{F} is thought of as both a formal power series and a linear functional. We call \mathcal{F} the umbral algebra. The umbral calculus is the study of umbral algebra (see[7, 11, 19, 20, 21]). The order $O(f(t))$ of the non-zero power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish (see[10, 20, 21]). If $O(f(t)) = 1$, then $f(t)$ is called a delta series and a series $f(t)$ having $O(f(t)) = 0$ is called an invertible series (see[10, 20, 21]). Let $f(t)$ be a delta series and $g(t)$ be an invertible series. Then there exists a unique sequence $S_n(x)$ of polynomials such that $\langle g(t)f(t)^k \mid S_n(x) \rangle = n! \delta_{n,k}$ where $n, k \geq 0$. The sequence $S_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$ which is denoted by $S_n(x) \sim (g(t), f(t))$. If $S_n(x) \sim (1, f(t))$, then $S_n(x)$ is called the associated sequence for $f(t)$. By (7), we see that $\langle e^{yt} \mid p(x) \rangle = p(y)$. For $f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$(8) \quad f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t) \mid x^k \rangle}{k!} t^k, \quad p(x) = \sum_{k=0}^{\infty} \frac{\langle t^k \mid p(x) \rangle}{k!} x^k, \quad (\text{see}[11, 20, 21]).$$

and

$$(9) \quad \begin{aligned} & \langle f_1(t)f_2(t)\dots f_m(t) \mid x^n \rangle \\ &= \sum_{i_1+\dots+i_m=n} \binom{n}{i_1, \dots, i_m} \langle f_1(t) \mid x^{i_1} \rangle \dots \langle f_m(t) \mid x^{i_m} \rangle, \end{aligned}$$

where $f_1(t), f_2(t), \dots, f_m(t) \in \mathcal{F}$.

By (8), we get

$$(10) \quad p^{(k)}(0) = \langle t^k \mid p(x) \rangle, \quad \langle 1 \mid p^{(k)}(x) \rangle = p^{(k)}(0).$$

Thus, by (10), we see that

$$(11) \quad t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}, \quad (k \geq 0), \quad (\text{see}[10, 20, 21]).$$

For $S_n(x) \sim (g(t), f(t))$, we have

$$(12) \quad \frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} \frac{S_k(y)}{k!} t^k, \quad \text{for all } y \in \mathbb{C},$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$,

$$(13) \quad S_n(x+y) = \sum_{k=0}^n \binom{n}{k} p_k(y) S_{n-k}(x) = \sum_{k=0}^n \binom{n}{k} S_{n-k}(y) p_k(x),$$

where $p_k(y) = g(t)S_k(y) \sim (1, f(t))$, (see [5, 11, 20, 21]).

Let $p_n(x) \sim (1, f(t))$ and $q_n(x) \sim (1, g(t))$. Then the transfer formula for the associated sequence implies that

$$(14) \quad q_n(x) = x \left(\frac{f(t)}{g(t)} \right)^n x^{-1} p_n(x), \quad (\text{see [5, 11, 20, 21]}).$$

In this paper, we investigate some properties of the associated sequence of Daehee and Changhee polynomials arising from umbral calculus. Finally, we derive some interesting identities of associated sequence of special polynomials from (14).

2. On the associated sequence of special polynomials

For $\lambda \in \mathbb{C}$ with $\lambda \neq 1$, the Daehee polynomials are defined by the generating function to be

$$(15) \quad \sum_{k=0}^{\infty} \frac{D_k(x|\lambda)}{k!} t^k = \left(\frac{(1-\lambda) + t(1+\lambda)}{(1-\lambda)(1-t)} \right) \left(\frac{1+t}{1-t} \right)^x,$$

(see [1, 2, 14, 16, 17, 18]).

From (15), we note that $D_n(x|\lambda)$ is the Sheffer sequence for $\left(\frac{1-\lambda}{e^t-\lambda}, \frac{e^t-1}{e^t+1} \right)$.

That is,

$$(16) \quad D_n(x|\lambda) \sim \left(\frac{1-\lambda}{e^t-\lambda}, \frac{e^t-1}{e^t+1} \right).$$

As is known, the Mittag-Leffler sequence is given by

$$(17) \quad M_n(x) = \sum_{k=0}^n \binom{n}{k} (n-1)_{n-k} 2^k (x)_k \sim \left(1, \frac{e^t-1}{e^t+1} \right).$$

From (16) and (17), we have

$$(18) \quad M_n(x) = \frac{1-\lambda}{e^t-\lambda} D_n(x|\lambda) \sim \left(1, \frac{e^t-1}{e^t+1} \right).$$

Let us consider the following associated sequence:

$$(19) \quad S_n(x|\lambda) \sim \left(1, \frac{1-\lambda}{e^t-\lambda} t \right) \quad \text{and} \quad x^n \sim (1, t).$$

By (14) and (19), we get

$$\begin{aligned}
 (20) \quad S_n(x|\lambda) &= x \left(\frac{\frac{1-\lambda}{e^t-\lambda}t}{t} \right)^{-n} x^{-1} x^n \\
 &= (1-\lambda)^{-n} x (e^t - \lambda)^n x^{n-1} \\
 &= (1-\lambda)^{-n} x \sum_{l=0}^n \binom{n}{l} (-\lambda)^{n-l} e^{lt} x^{n-1} \\
 &= (1-\lambda)^{-n} x \sum_{l=0}^n \binom{n}{l} (-\lambda)^{n-l} (x+l)^{n-1}.
 \end{aligned}$$

Therefore by (20), we obtain the following theorem.

Theorem 1 . For $r \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, let $S_n(x|\lambda) \sim (1, \frac{1-\lambda}{e^t-\lambda}t)$. Then we have

$$S_n(x|\lambda) = \frac{x}{(1-\lambda)^n} \sum_{l=0}^n \binom{n}{l} (-\lambda)^{n-l} (x+l)^{n-1}.$$

From (14), (17) and (19), we have

(21)

$$\begin{aligned}
 M_n(x) &= x \left(\frac{\frac{1-\lambda}{e^t-\lambda}t}{\frac{e^t-1}{e^t+1}} \right)^n x^{-1} S_n(x) = x \left(\frac{1-\lambda}{e^t-\lambda} \right)^n \left(\frac{t(e^t+1)}{e^t-1} \right)^n x^{-1} S_n(x) \\
 &= x \left(\frac{1-\lambda}{e^t-\lambda} \right)^n \left(t + \frac{2t}{e^t-1} \right)^n x^{-1} S_n(x) \\
 &= x \left(\frac{1-\lambda}{e^t-\lambda} \right)^n \sum_{l=0}^n \binom{n}{l} t^{n-l} \left(\frac{2t}{e^t-1} \right)^l x^{-1} S_n(x) \\
 &= x \left(\frac{1-\lambda}{e^t-\lambda} \right)^n \sum_{l=0}^n \binom{n}{l} t^{n-l} \left(\frac{2t}{e^t-1} \right)^l \frac{1}{(1-\lambda)^n} \sum_{j=0}^n \binom{n}{j} (-\lambda)^{n-j} (x+j)^{n-1} \\
 &= x \left(\frac{1-\lambda}{e^t-\lambda} \right)^n \sum_{l=0}^n \sum_{j=0}^n \binom{n}{l} (1-\lambda)^{-n} \binom{n}{j} (-\lambda)^{n-j} t^{n-l} \left(\frac{2t}{e^t-1} \right)^l (x+j)^{n-1}.
 \end{aligned}$$

By (1), (2) and (12), we get

$$(22) \quad B_n^{(\alpha)}(x) \sim \left(\left(\frac{e^t-1}{t} \right)^\alpha, t \right) \quad H_n^{(\alpha)}(x|\lambda) \sim \left(\left(\frac{e^t-\lambda}{1-\lambda} \right)^\alpha, t \right),$$

and, by (13), we see that

$$(23) \quad B_n^{(\alpha)}(x) = \sum_{k=0}^n \binom{n}{k} B_{n-k}^{(\alpha)} x^k, \quad H_n^{(\alpha)}(x|\lambda) = \sum_{l=0}^n \binom{n}{l} H_{n-l}^{(\alpha)} x^l.$$

Thus, by (21), (22) and (23), we get

$$(24)$$

$$\begin{aligned} M_n(x) &= x \left(\frac{1-\lambda}{e^t-\lambda} \right)^n \sum_{l=0}^n \sum_{j=0}^n \binom{n}{l} (1-\lambda)^{-n} \binom{n}{j} (-\lambda)^{n-j} 2^l t^{n-l} B_{n-1}^{(l)}(x+j) \\ &= x \left(\frac{1-\lambda}{e^t-\lambda} \right)^n \sum_{l=1}^n \sum_{j=0}^n \binom{n}{l} (1-\lambda)^{-n} \binom{n}{j} (-\lambda)^{n-j} 2^l (n-1)_{n-l} B_{l-1}^{(l)}(x+j) \\ &= x \left(\frac{1-\lambda}{e^t-\lambda} \right)^n \sum_{l=1}^n \sum_{j=0}^n \binom{n}{l} (1-\lambda)^{-n} \binom{n}{j} (-\lambda)^{n-j} 2^l \frac{(n-1)!}{(l-1)!} \sum_{m=0}^{l-1} B_{l-1-m}^{(l)} \\ &\quad \times \binom{l-1}{m} (x+j)^m \\ &= x \sum_{l=1}^n \sum_{j=0}^n \binom{n}{l} (1-\lambda)^{-n} \binom{n}{j} (-\lambda)^{n-j} 2^l \frac{(n-1)!}{(l-1)!} \sum_{m=0}^{l-1} B_{l-1-m}^{(l)} \binom{l-1}{m} \\ &\quad \times \left(\frac{1-\lambda}{e^t-\lambda} \right)^n (x+j)^m \\ &= \frac{x(n-1)!}{(1-\lambda)^n} \sum_{l=1}^n \sum_{j=0}^n \sum_{m=0}^{l-1} \frac{\binom{n}{l} \binom{n}{j} \binom{l-1}{m} (-\lambda)^{n-j} 2^l}{(l-1)!} B_{l-1-m}^{(l)} H_m^{(n)}(x+j|\lambda). \end{aligned}$$

Recall here that, for any $g(t) \in \mathcal{F}$, the Pincherle derivative is given by

$$(25) \quad g'(t) = g(t)x - xg(t)$$

as linear operators on \mathbb{P} , (see [22]).

By (18), (24) and (25), we get

$$\begin{aligned} (26) \quad D_n(x|\lambda) &= \frac{e^t-\lambda}{1-\lambda} M_n(x) \\ &= \frac{(n-1)!}{(1-\lambda)^n} \sum_{l=1}^n \sum_{j=0}^n \sum_{m=0}^{l-1} \frac{\binom{n}{l} \binom{n}{j} \binom{l-1}{m} (-\lambda)^{n-j} 2^l}{(l-1)!} B_{l-1-m}^{(l)} \\ &\quad \times \left\{ x H_m^{(n-1)}(x+j|\lambda) + \frac{1}{1-\lambda} H_m^{(n)}(x+j+1|\lambda) \right\}. \end{aligned}$$

Therefore, by (26), we obtain the following theorem.

Theorem 2 . For $n \geq 1$, we have

$$\begin{aligned} D_n(x|\lambda) &= \frac{(n-1)!}{(1-\lambda)^n} \sum_{l=1}^n \sum_{j=0}^n \sum_{m=0}^{l-1} \frac{\binom{n}{l} \binom{n}{j} \binom{l-1}{m} (-\lambda)^{n-j} 2^l}{(l-1)!} B_{l-1-m}^{(l)} x \\ &\quad \times \left\{ x H_m^{(n-1)}(x+j|\lambda) + \frac{1}{1-\lambda} H_m^{(n)}(x+j+1|\lambda) \right\}. \end{aligned}$$

Remark . (a) Let us consider the following associated sequence for $\frac{t}{e^t+1}$:

$$(27) \quad S_n(x) \sim \left(1, \frac{t}{e^t+1}\right).$$

By (27), we get

$$(28) \quad S_n(x) = x(e^t+1)^n x^{-1} x^n = x \sum_{j=0}^n \binom{n}{j} (x+j)^{n-1}.$$

From (14), (17) and (27), we have

$$(29) \quad M_n(x) = x \left(\frac{t}{e^t-1} \right)^n x^{-1} S_n(x) = \sum_{j=0}^n \binom{n}{j} x B_{n-1}^{(n)}(x+j).$$

Thus, by (18) and (29), we get

$$\begin{aligned} (30) \quad D_n(x|\lambda) &= \left(\frac{e^t - \lambda}{1 - \lambda} \right) M_n(x) \\ &= \frac{1}{1-\lambda} \sum_{j=0}^n \binom{n}{j} (x+1) B_{n-1}^{(n)}(x+j+1) + \frac{\lambda}{\lambda-1} \sum_{j=0}^n \binom{n}{j} x B_{n-1}^{(n)}(x+j). \end{aligned}$$

(b) From (17), we note that we can obtain another expression of $D_n(x|\lambda)$ as follow :

$$\begin{aligned} (31) \quad D_n(x|\lambda) &= \left(\frac{e^t - \lambda}{1 - \lambda} \right) M_n(x) \\ &= \sum_{k=0}^n \binom{n}{k} (n-1)_{n-k} 2^k \frac{e^t - \lambda}{1 - \lambda} (x)_k \\ &= \frac{1}{1-\lambda} \sum_{k=0}^n \binom{n}{k} (n-1)_{n-k} 2^k \{ (x+1)_k - \lambda (x)_k \}. \end{aligned}$$

For $\lambda \in \mathbb{C}$ with $\lambda \neq 1$, Changhee polynomials of order a are defined by the generating function to be

$$(32) \quad \sum_{k=0}^{\infty} C_k^{(a)}(x|\lambda) \frac{t^k}{k!} = \left(\frac{t+1-\lambda}{1-\lambda} \right)^a (1+t)^x, \quad (\text{see } [15, 23, 24, 25, 26]).$$

In the special case, $x = 0$, $C_k^{(a)}(0|\lambda) = C_k^{(a)}(\lambda)$ are called n -th Changhee numbers of order a . From (12) and (21), we note that

$$(33) \quad C_n^{(a)}(x|\lambda) \sim \left(\left(\frac{1-\lambda}{e^t-\lambda} \right)^a, e^t - 1 \right), \quad a \neq 0.$$

Thus, by (33), we get

$$(34) \quad \left(\frac{e^t - \lambda}{1 - \lambda} \right)^a C_n^{(a)}(x|\lambda) \sim (1, e^t - 1).$$

From (14), $x^n \sim (1, t)$ and (35), we have

$$(35) \quad \begin{aligned} \left(\frac{e^t - \lambda}{1 - \lambda} \right)^a C_n^{(a)}(x|\lambda) &= x \left(\frac{t}{e^t - 1} \right)^n x^{-1} x^n \\ &= x \left(\frac{t}{e^t - 1} \right)^n x^{n-1} = x B_{n-1}^{(n)}(x). \end{aligned}$$

Thus, by (14), (22), (25) and (35), we get

$$(36) \quad \begin{aligned} C_n^{(a)}(x|\lambda) &= \left\{ x \left(\frac{1-\lambda}{e^t-\lambda} \right)^a - \frac{a}{1-\lambda} e^t \left(\frac{1-\lambda}{e^t-\lambda} \right)^{a+1} \right\} B_{n-1}^{(n)}(x) \\ &= \sum_{l=0}^{n-1} \binom{n-1}{l} B_l^{(n)} \left\{ x \left(\frac{1-\lambda}{e^t-\lambda} \right)^a - \frac{a}{1-\lambda} e^t \left(\frac{1-\lambda}{e^t-\lambda} \right)^{a+1} \right\} x^{n-1-l} \\ &= \sum_{l=0}^{n-1} \binom{n-1}{l} B_l^{(n)} \left\{ x H_{n-1-l}^{(a)}(x|\lambda) - \frac{a}{1-\lambda} H_{n-1-l}^{(a+1)}(x+1|\lambda) \right\}. \end{aligned}$$

Therefore, by (36), we obtain the following theorem.

Theorem 3 . For $n \in \mathbb{N}$, we have

$$C_n^{(a)}(x|\lambda) = \sum_{l=0}^{n-1} \binom{n-1}{l} B_l^{(n)} \left\{ x H_{n-1-l}^{(a)}(x|\lambda) - \frac{a}{1-\lambda} H_{n-1-l}^{(a+1)}(x+1|\lambda) \right\}.$$

Let

$$(37) \quad p(x) \sim \left(1, t \left(\frac{e^t - \lambda}{1 - \lambda} \right)^a \right).$$

By $x^n \sim (1, t)$ and (14), we get

$$(38) \quad \begin{aligned} p(x) &= x \left(\frac{t}{t \left(\frac{e^t - \lambda}{1 - \lambda} \right)^a} \right)^n x^{-1} x^n \\ &= x \left(\frac{1 - \lambda}{e^t - \lambda} \right)^{an} x^{n-1} = x H_{n-1}^{(an)}(x|\lambda), \end{aligned}$$

where $a \in \mathbb{Z}_+$.

Therefore, by (38), we obtain the following theorem.

Theorem 4 . For $n \in \mathbb{N}$ and $a \in \mathbb{Z}_+$, we have

$$xH_{n-1}^{(an)}(x|\lambda) \sim \left(1, t \left(\frac{1-\lambda}{e^t-\lambda}\right)^a\right).$$

By the same method of (38), we easily see that

$$(39) \quad xB_{n-1}^{(bn)}(x) \sim \left(1, t \left(\frac{e^t-1}{t}\right)^b\right), \quad b \in \mathbb{Z}_+.$$

For $n \geq 1$, by (14), Theorem 4 and (39), we get

$$(40) \quad \begin{aligned} xH_{n-1}^{(an)}(x|\lambda) &= x \left(\frac{t \left(\frac{e^t-1}{t}\right)^b}{t \left(\frac{e^t-\lambda}{1-\lambda}\right)^a} \right)^n x^{-1} xB_{n-1}^{(an)}(x) \\ &= x \left(\frac{1-\lambda}{e^t-\lambda}\right)^{an} \left(\frac{e^t-1}{t}\right)^{bn} B_{n-1}^{(an)}(x). \end{aligned}$$

Thus, from (40), we have

$$(41) \quad \frac{(e^t-\lambda)^{an}}{(1-\lambda)^{an}} H_{n-1}^{(an)}(x|\lambda) = \left(\frac{e^t-1}{t}\right)^{bn} B_{n-1}^{(bn)}(x).$$

$$(42) \quad \begin{aligned} LHS \text{ of } (40) &= \frac{1}{(1-\lambda)^{an}} \sum_{j=0}^{an} \binom{an}{j} (-\lambda)^{an-j} e^{jt} H_{n-1}^{(an)}(x|\lambda) \\ &= \frac{1}{(1-\lambda)^{an}} \sum_{j=0}^{an} \binom{an}{j} (-\lambda)^{an-j} H_{n-1}^{(an)}(x+j|\lambda), \end{aligned}$$

and

$$\begin{aligned}
(43) \quad RHS \text{ of } (41) &= \left(\frac{e^t - 1}{t} \right)^{bn} B_{n-1}^{(bn)}(x) \\
&= \frac{1}{t^{bn}} (bn)! \sum_{j=bn}^{\infty} S_2(j, bn) \frac{t^j}{j!} B_{n-1}^{(bn)}(x) \\
&= (bn)! \sum_{j=0}^{n-1} \frac{S_2(j + bn, bn)}{(j + bn)!} t^j B_{n-1}^{(bn)}(x) \\
&= \sum_{j=0}^{n-1} \frac{(bn)!}{(j + bn)!} \frac{(n-1)!}{(n-j-1)!} S_2(j + bn, bn) B_{n-1-j}^{(bn)}(x) \\
&= (n-1)! \sum_{j=0}^{n-1} \frac{(bn)!}{(j + bn)!} \frac{S_2(j + bn, bn)}{(n-j-1)!} B_{n-1-j}^{(bn)}(x).
\end{aligned}$$

Therefore, by (41), (42) and (43), we obtain the following theorem.

Theorem 5 . For $a, b \in \mathbb{Z}_+$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\sum_{j=0}^{an} \binom{an}{j} (-\lambda)^{an-j} H_{n-1}^{(an)}(x + j|\lambda) \\
&= (1 - \lambda)^{an} (n-1)! \sum_{j=0}^{n-1} \frac{(bn)!}{(j + bn)! (n-j-1)!} S_2(j + bn, bn) B_{n-1-j}^{(bn)}(x).
\end{aligned}$$

Remark . Let us consider the following Sheffer sequence:

$$(44) \quad S_n(x|\lambda) = \left(\frac{1 - \lambda}{e^t - \lambda}, t \left(\frac{1 - \lambda}{e^t - \lambda} \right) \right).$$

Thus, by (44), we get

$$(45) \quad \frac{1 - \lambda}{e^t - \lambda} S_n(x|\lambda) \sim \left(1, t \left(\frac{1 - \lambda}{e^t - \lambda} \right) \right).$$

By Theorem 5 and (45), we get

$$\begin{aligned}
S_n(x|\lambda) &= \left(\frac{e^t - \lambda}{1 - \lambda} \right) x H_{n-1}^{(n)}(x|\lambda) \\
&= \frac{1}{1 - \lambda} \{ (x + 1) H_{n-1}^{(n)}(x + 1|\lambda) - \lambda x H_{n-1}^{(n)}(x|\lambda) \}.
\end{aligned}$$

The Daehee polynomials of the second kind are defined by the generating function to be

$$(46) \quad \sum_{k=0}^{\infty} D_k^*(x|\lambda) \frac{t^k}{k!} = \frac{1}{1 - t} \left(\frac{1 - \lambda t}{1 - t} \right)^x,$$

where $\lambda \in \mathbb{C}$ with $\lambda \neq 1$, (see [2, 14, 26, 28]).

From (12) and (46), we note that

$$(47) \quad D_n^*(x|\lambda) \sim \left(\frac{1-\lambda}{e^t-\lambda}, \frac{e^t-1}{e^t-\lambda} \right).$$

Let us consider the λ -analogues of Mittag-Leffler sequence as follows:

$$(48) \quad M_n^*(x|\lambda) \sim \left(1, \frac{e^t-1}{e^t-\lambda} \right).$$

From (14), $x^n \sim (1, t)$ and (48), we have

$$(49) \quad \begin{aligned} M_n^*(x|\lambda) &= x \left(\frac{e^t-\lambda}{e^t-1} t \right)^n x^{n-1} = x(e^t-\lambda)^n \left(\frac{t}{e^t-1} \right)^n x^{n-1} \\ &= x \sum_{l=0}^n \binom{n}{l} (-\lambda)^{n-l} e^{lt} B_{n-1}^{(n)}(x) \\ &= x \sum_{l=0}^n \binom{n}{l} (-\lambda)^{n-l} B_{n-1}^{(n)}(x+l). \end{aligned}$$

Therefore, by (49), we obtain the following theorem.

Theorem 6 . For $n \in \mathbb{N}$, let $M_n^*(x|\lambda) \sim \left(1, \frac{e^t-1}{e^t-\lambda} \right)$

Then we have

$$M_n^*(x|\lambda) = \sum_{l=0}^n \binom{n}{l} (-\lambda)^{n-l} x B_{n-1}^{(n)}(x+l).$$

From (13) and Theorem 6, we have

$$(50) \quad M_n^*(x|\lambda) = x \sum_{l=0}^n \sum_{j=0}^{n-1} \binom{n}{l} \binom{n-1}{j} (-\lambda)^{n-l} B_{n-1-j}^{(n)}(x+l)^j.$$

By (47) and (48), we get

$$(51) \quad \begin{aligned} D_n^*(x|\lambda) &= \left(\frac{e^t-\lambda}{1-\lambda} \right) M_n^*(x|\lambda) \\ &= \frac{1}{1-\lambda} \sum_{j=0}^n \binom{n}{j} \{ (x+1) B_{n-j}^{(n)}(x+1+j) - \lambda x B_{n-1}^{(n)}(x+j) \} (-\lambda)^{n-j}. \end{aligned}$$

Let us consider the following associated sequences:

$$(52) \quad T_n^*(x) \sim \left(1, \frac{2t}{1+t^2} \right).$$

From (12) and (52), we have

$$(53) \quad \sum_{k=0}^{\infty} T_k^*(x) \frac{t^k}{k!} = \exp \left(x \left(\frac{1 - \sqrt{1 - t^2}}{t} \right) \right).$$

By (14), (52) and $x^n \sim (1, t)$, we get

$$\begin{aligned} T_n^*(x) &= x \left(\frac{t}{\frac{2t}{1+t^2}} \right)^n x^{-1} x^n = x \left(\frac{1+t^2}{2} \right)^n x^{n-1} \\ &= \left(\frac{1}{2} \right)^n x \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{l} t^{2l} x^{n-1} \\ &= \left(\frac{1}{2} \right)^n \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{l} \frac{(n-1)!}{(n-2l-1)!} x^{n-2l}. \end{aligned}$$

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